Measure theory

- The goal is to define the notion of probability
- We define slightly more general notion of measure

σ-algebra (sigma-algebra)

Definition: Let X be a set. A set system $F \subseteq \mathcal{P}(X)$ is called a σ-algebra iff it satisfies three conditions

1) $X \in F$

2) If $A \in F$ then $\bar{A} \in F$

(closure under complements)

3) If $A_1, A_2 \ldots \in F$ then

$$\left(\bigcup_{i=1}^{\infty} A_i \right) \in F$$

(closure under countable unions)

Examples of σ-algebras

- $F = \mathcal{P}(X)$
- $F = \{\emptyset, X\}$
- If X is finite or countable, we almost always take $F = \mathcal{P}(X)$
- If $X = \mathbb{R}$ we usually work

with so called <u>Borel σ-algebra</u>.

- It is the "smallest" σ-algebra containing all closed intervals.
- It contains all closed, all open intervals, unions of countably many intervals, and many other types of sets.
- It does <u>NOT</u> contain all subsets of $\mathbb{R}$

<u>Measure</u>

- Measure is generalization of notions of <u>length</u>, <u>area</u>, <u>volume</u> and <u>probability</u>.

__Definition__: Let F be a σ-algebra over set X. A __measure__ is a function $\mu : F \to \mathbb{R} \cup \{\infty\}$ satisfying three conditions:

1) $\mu(A) \geq 0$ for all $A \in F$

   (non-negativity)

2) $\mu(\emptyset) = 0$

   (null empty set)

3) If $A_1, A_2, \ldots$ are pairwise __disjoint__ then

∞

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1} \mu(A_i)$$

(countable additivity)

Examples of measures

- $\mu(A) = |A|$ (counting measure)

- $\mu([a,b]) = b-a$ (Lebesgue measure on $\mathbb{R}$)

Probability

probability = probability measure

= probability distribution

= distribution

These mean the same thing.

<u>Definition</u>: Let F be a σ-algebra over X. A probability measure $\mathcal{D}$ is a measure such that

$\mathcal{D}(X) = 1$.

<u>Examples of probability measures</u>

- Uniform distribution over a <u>finite</u> set X:

$$\mathcal{D}(A) = \frac{|A|}{|X|}$$

- Non-uniform discrete probability distribution over <u>finite</u> or <u>countable</u> set X:

$$D(A) = \sum_{a \in A} p(a) \quad \text{(lower-case } p\text{)}$$

where $p: X \to \mathbb{R}$ satisfies

1) $p(a) \geq 0$

2) $\displaystyle\sum_{a \in X} p(a) = 1$

- Continuous measure over $\mathbb{R}$

$$D(A) = \int_A f(x)\, dx$$

where $f : \mathbb{R} \to \mathbb{R}$ is integrable function such that

1) $f(x) \geq 0$ for all $x \in \mathbb{R}$

2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

- There many more probability distributions

Notation

- If D is a probability distribution,

we write

$$\Pr[A] = D(A).$$

An element $A \in F$ is called an <u>event</u> or <u>probability event</u>.

- Usually σ-algebra has to be understood from context. It is often not explicitly specified.
- Instead of letter D often letters P or μ are used.